UPSC Mathematics Optional

Previous Year Question Solutions

COMPLEX ANALYSIS

2017

Q1. Let $f = u + iv$ be analytic function on the unit disc $D = \{z \in c : |z| < 1\}$.

Show that $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ at all points of D

[15 Marks]

Solution:

Given that the function

$f(z) = 4(x, y) + iv(x, y)$ is analytic in a domain

$\therefore f'(z) = \frac{\partial f}{\partial x}$ ...(i)

or $f'(z) = -i\frac{\partial f}{\partial y}$ ...(ii)

Since the analytic function has derivatives of all orders

$$(1) \equiv f''(z) = \frac{\partial}{\partial x}\left(f'(z)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} \quad \text{...(iii)}$$

$$(2) \equiv f''(z) = -i\frac{\partial}{\partial y}\left(f'(z)\right)$$

$$= -i\frac{\partial}{\partial y}\left(-i\frac{\partial f}{\partial y}\right)$$

$$= (-1)\frac{\partial^2 f}{\partial y^2} \quad \text{...(iv)}$$

for (iii) & (iv) we have,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 f = 0 \quad \text{...}(v)$$

Which is valid for any analytic function $f(z)$

i.e. If $f(z) = 4(x, y) + iv(x, y)$ is an analytic in a domain D then from v)

$$\nabla^2 f = 0$$

$$\Rightarrow \nabla^2(u+iv)=0$$
$$\Rightarrow \nabla^2 u + i\nabla^2 v = 0$$
$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0; \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Q2. Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f(1/2)$?

[10 Marks]

Solution:

As the function $f(z)$ has no singularity in the finite part of the plane; it can be expanded in Taylor's series in any circle $|z|=k$; where k is arbitrary large.

$$\therefore f(z)=\sum_{r=0}^{\infty} A_r z^r$$

Also, if $f(z)$ has no -singularity at $z=\infty$; $f(1/2)$ has none at $z=0$.

Moreover; since $f(z)$ has no singularity in finite part of plane, we have,

$$f(z)=\sum_{r=0}^{\infty} A_r z^r, \qquad f(1/z)=\sum_{r=0}^{\infty} A_r z^{-r}$$

Since, $f(2)$ has a pole of order 'n' at infinity $f(1/2)$ has a pole of order n at zero.

$$\therefore f(1/z)=\sum_{s=1}^{n}\frac{B_s}{z^s}+\phi(z)$$

Where, $\phi(z)$ is a regular function of z.

Hence, $\phi(z)+\sum_{s=1}^{n}\frac{B_s}{z^s}=\sum_{r=0}^{\infty} A_r z^{-r}$

$\therefore \phi(z)=A_0=a$ constant .

and $A_{n+1}=A_{n+2}=\cdots=0$

$$f(1/z)=\sum_{\gamma=0}^{\infty} A_n z^{-\gamma}=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots$$

$f\left(\frac{1}{2}\right)$ is a removable singularity of degree r.

Q3. Using Contour integral method, prove that

$$\int_0^{\infty}\frac{x\sin mx}{a^2+x^2}dx=\frac{\pi}{2}e^{-ma}$$

[15 Marks]

Solution:

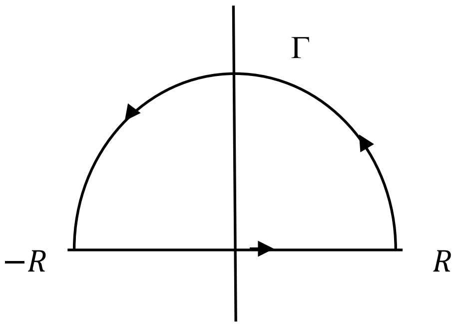
Let

$$I=\int_C \frac{e^{imz}}{a^2+z^2}dz=\int_C f(z)dx,$$

where C is the contour consisting of a large semicircle, T of orders R containing all the poles of the integrand in the upper half plane and the part of real axis from $-R$ to R.

By Cauchy-Residue Theorem:

$$\int_C f(z)dz = \int_{-R}^{R} \frac{e^{imx}}{a^2+x^2}dx + \int_T \frac{e^{imz}}{a^2+z^2}dz$$
$$= 2\pi i \qquad \text{(Sum of Residues)}$$



Since, $\lim_{z\to\infty} \frac{1}{\left(z^2+a^2\right)} = 0$, therefore

$\lim_{R\to\infty} \int_T f(z)dz = 0$, by Jordan Lemma

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{imx}}{a^2+x^2}dx = 2\pi i \qquad \text{(Sure of Residues)}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2}dx = 2\pi i \qquad \text{(Sum of Residue)} \qquad \ldots(1)$$

$z = \pm ai$ are simple poles of $f(z)$.

The pole, $z = ai$ lies inside, C. Residue at $z = ai$ is

$$\lim_{z\to ai}(z-ai)f(z) = \lim_{z\to ai} \frac{(z-ai)e^{imz}}{a^2+z^2}$$
$$= \lim_{z\to ai} \frac{e^{imz}}{z+ai} = \frac{e^{-ma}}{2ia}$$

from (1)

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx = 2\pi i\left(\frac{e^{-ma}}{2ia}\right) = \frac{\pi}{a}e^{-ma}$$

Equating real parts on both sides,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

or $$\int_0^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma}$$

Differentiating both sides w.r.t. m

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \frac{\pi}{2}e^{-ma}.$$

Q4. For a function $f:\Phi \to \mathcal{C}$ and $n \geq 1$, let f^n denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \geq 1$, $f^n(1/k) = 0$ for all $k = 1,2,3,\ldots$. show that f is a polynomial.

[15 Marks]

Solution:

Since $f(z)$ is entire, $\therefore f(z)$ is analytic.

Hence, $f(z)$ can be expressed as Taylor's series around $z = 0$ as

$$f(z) = \sum_{m=1}^{\infty} a_{-m}\frac{1}{z^m} + \sum_{m=0}^{\infty} a_m z^m$$

$$f^n(z) = \sum_{m=1}^{\infty} (-1)^n(m)(m+1)\ldots(m+1(n-1))a_{-m}\cdot\frac{1}{z^{m+n}} + \sum_{m=n}^{\infty} m(m-1)\ldots(m-(n-1))a_m\cdot z^{m-n}$$

Now,

$$f^n\left(\frac{1}{k}\right)=\sum_{m=1}^{\infty}(-1)^m(m)(m+1)\cdots(m+(n-1))a_{-m}\cdot k^{m+n}+\sum_{m=n}^{\infty}m(m-1)\ldots(m-n+1)a_m\left(\frac{1}{k}\right)^{m-n}$$

As, $f^n\left(\frac{1}{k}\right)\to 0$ for all $k=1,2,3,\ldots$

Let us take $k\to\infty$

We get $a_{-n}\to 0$ and $a_n=0$

Similarly, if we take $m+1, a_{n+1}=0$

$\therefore a_n=a_{n+1}=a_{n+2}=\ldots=0$

$\therefore f(z)=a_0+a_1z+a_2z^2+\cdots+a_{n-1}z^{n-1}$

which is a polynomial function.

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